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A Linear Scheme for Rational Approximations*

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1. INTRODUCTION

If a function $F(v)$ can be represented on a region Ω in the form

$$F(v) = \sum_{j=0}^{k} f_j v^{-j} + R_{k+1}(v), \qquad k = 0, 1,..., \tag{1.1}$$

where the f_j are independent of v and k, then a formal rational approximation to $F(v)$ can be derived as follows. For fixed n, let the weighting coefficients $A_{n,k}\gamma^k$, $k = 0, 1,..., n$, be arbitrary. Replacing k by $k - a$ in (1.1), a an integer $\geqslant 0$, multiplying the resulting equation by $A_{n,k}\gamma^k$, and summing from $k = 0$ to n, one obtains

$$F(v)\, H_n{}^a(\gamma) = K_n{}^a(v, \gamma) + S_n{}^a(v, \gamma), \tag{1.2}$$

$$H_n{}^a(\gamma) = \sum_{k=0}^{n} A_{n,k}\gamma^k,$$

$$S_n{}^a(v, \gamma) = \sum_{k=0}^{n} A_{n,k}\gamma^k R_{k+1-a}(v),$$

$$K_n{}^a(v, \gamma) = \sum_{k=a}^{n} \gamma^k \sum_{r=0}^{n-k} A_{n,k+r} f_r (\gamma/v)^r. \tag{1.3}$$

Then $K_n{}^a(v, \gamma)/H_n{}^a(\gamma)$ is the desired formal rational approximation to $F(v)$. Although the range parameter γ is almost always chosen equal to v, there are advantages in leaving it arbitrary as long as possible.

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If $\mathscr{M}_n{}^a(\gamma)$ is a linear difference operator with respect to n which annihilates both $H_n{}^a(\gamma)$ and $K_n{}^a(v, \gamma)$, the study of the convergence properties of the sequence $\{K_n{}^a(v, \gamma)/H_n{}^a(\gamma)\}_{n=0}^{\infty}$ reduces essentially to an analysis of the difference equation $\mathscr{M}_n{}^a(\gamma)\{y_n(\gamma)\} = 0$.

The parameter a distinguishes between classes of rational approximations that correspond to the main and off diagonal entries of the classical Padé table ([1]). As the parameter a will be fixed, its dependence will usually be suppressed.

Using this scheme, explicit rational approximations $K_n(v)/H_n(v)$ are developed for the Meijer G-functions (generalized hypergeometric functions [2, 3])

$$\begin{aligned} F(v) &\equiv \frac{\prod_{k=1}^{q} \Gamma(\rho_k)}{\prod_{r=1}^{p} \Gamma(\alpha_r)} G_{p,q+1}^{1,p}\left(v^{-1} \middle| \begin{matrix} 1-\alpha_1, ..., 1-\alpha_p \\ 0, 1-\rho_1, ..., 1-\rho_q \end{matrix}\right), \quad \beta = p + 1 - q > 2, \\ &= \frac{1}{2\pi i} \int_{L_-} \frac{\Gamma(-s) \prod_{r=1}^{p} (\alpha_r)_s \, v^{-s}}{\prod_{k=1}^{q} (\rho_k)_s} \, ds, \qquad (\sigma)_s \equiv \frac{\Gamma(s+\sigma)}{\Gamma(\sigma)}, \\ &\sim {}_pF_q\left(\begin{matrix} \alpha_1, ..., \alpha_p \\ \rho_1, ..., \rho_q \end{matrix} \middle| \frac{-1}{v}\right), \qquad v \to \infty, \qquad |\arg v| < \pi\beta/2, \end{aligned} \tag{1.4}$$

where L_- is a loop contour running from $\infty e^{-i\pi}$ to $\infty e^{i\pi}$ which separates the poles of $\prod_{r=1}^{p} \Gamma(s + \alpha_r)$ from those of $\Gamma(-s)$. Under mild restrictions on the parameters a, α_r and ρ_k, it will be shown that

(1) $F(v) = \lim_{n\to\infty} K_n(v)/H_n(v)$, $v(\neq 0)$ fixed, $|\arg v| < \pi$,

(2) as for the Padé approximants, $K_n(v)$ and $H_n(v)$ satisfy the same homogeneous difference equation with respect to n, and

(3) the error, $F(v) - K_n(v)/H_n(v)$, can be represented by an easily analyzed, closed form expression.

Partial results for the $\beta = 2, 3, 4$ cases are collected in [3, 4, and 5].

In what follows, I will make use of the notation

$$\begin{aligned} G_{p,q}^{m,n}\left(w \middle| \begin{matrix} a_1, ..., a_p \\ b_1, ..., b_q \end{matrix}\right) &= G_{p,q}^{m,n}\left(w \middle| \begin{matrix} a_P \\ b_Q \end{matrix}\right), \\ &= \frac{1}{2\pi i} \int_L \frac{\Gamma(b_M - t)\, \Gamma(1 - a_N + t)\, w^t}{\Gamma_m(1 - b_Q + t)\, \Gamma_n(a_P - t)} \, dt, \end{aligned} \tag{1.5}$$

where

$$\Gamma_n(c_P - t) \equiv \prod_{k=n+1}^{p} \Gamma(c_k - t), \qquad \Gamma(c_M - t) = \Gamma_0(c_M - t), \tag{1.6}$$

and L is an upward oriented loop contour which separates the poles of $\Gamma(b_M - t)$ from those of $\Gamma(1 - a_N + t)$ and which begins and ends at $+\infty$ ($L = L_+$) or $-\infty$ ($L = L_-$). The basic functional relationships for the G-function are then

$$G_{p,q}^{m,n}\left(w \,\middle|\, \begin{matrix} a_P \\ b_Q \end{matrix}\right) = G_{q,p}^{n,m}\left(w^{-1} \,\middle|\, \begin{matrix} 1 - b_Q \\ 1 - a_P \end{matrix}\right), \tag{1.7}$$

$$w^c\, G_{p,q}^{m,n}\left(w \,\middle|\, \begin{matrix} a_P \\ b_Q \end{matrix}\right) = G_{p,q}^{m,n}\left(w \,\middle|\, \begin{matrix} c + a_P \\ c + b_Q \end{matrix}\right). \tag{1.8}$$

If the poles of the integrand in (1.5), interior to L, are simple, then the G-function is a finite sum of hypergeometric functions, e.g.,

$$G_{p,q}^{m,n}\left(w \,\middle|\, \begin{matrix} a_P \\ b_Q \end{matrix}\right)$$

$$= \sum_{k=1}^{m} \frac{\Gamma^*(b_M - b_k)\,\Gamma(1 - a_N + b_k)\,w^{b_k}}{\Gamma_m(1 - b_Q + b_k)\,\Gamma_n(a_P - b_k)}\; {}_{p+1}F_q\left(\begin{matrix} 1, 1 - a_P + b_k \\ 1 - b_Q + b_k \end{matrix} \,\middle|\, (-1)^{p-m-n} w\right),$$

$$p < q \quad \text{or} \quad p = q \quad \text{and} \quad |w| < 1, \tag{1.9}$$

where

$$(b_M{}^*)_{-b_k} = \frac{\Gamma(b_k)\,\Gamma^*(b_M - b_k)}{\Gamma(b_M)} = \prod_{\substack{j=1 \\ j \neq k}}^{m} \frac{\Gamma(b_j - b_k)}{\Gamma(b_j)}, \tag{1.10}$$

and the notation of (1.5) has been used for hypergeometric functions.

Combining the above notations, we rewrite (1.4) in the form

$$\begin{aligned} F(v) &\equiv \frac{\Gamma(\rho_Q)}{\Gamma(\alpha_P)}\, G_{p,q+1}^{1,p}\left(v^{-1} \,\middle|\, \begin{matrix} 1 - \alpha_P \\ 0, 1 - \rho_Q \end{matrix}\right), \qquad \beta = p + 1 - q > 2, \\ &= \frac{1}{2\pi i} \int_{L_-} \frac{\Gamma(-s)(\alpha_P)_s\, v^{-s}}{(\rho_Q)_s}\, ds, \qquad (\sigma)_s \equiv \frac{\Gamma(s + \sigma)}{\Gamma(\sigma)}, \\ &\sim {}_pF_q\left(\begin{matrix} \alpha_P \\ \rho_Q \end{matrix} \,\middle|\, \frac{-1}{v}\right), \qquad v \to \infty, \qquad |\arg v| < \pi\beta/2. \end{aligned} \tag{1.11}$$

To see that the above scheme is applicable, one moves the contour L_- in (1.11) $k + 1$ units to the right to obtain the analog of (1.1),

$$F(v) = \sum_{j=0}^{k} \frac{(\alpha_P)_j\,(-v)^{-j}}{(\rho_Q)_j\, j!} + (-1)^{k+1} \frac{\Gamma(\rho_Q)}{\Gamma(\alpha_P)}\, G_{p+1,q+2}^{1,p+1}\left(v^{-1} \,\middle|\, \begin{matrix} k + 1, 1 - \alpha_P \\ k + 1, 0, 1 - \rho_Q \end{matrix}\right). \tag{1.12}$$

For convenience, we set $K_n{}^a(v, v)$ and $S_n{}^a(v, v)$ equal to $K_n(v)$ and $S_n(v)$, respectively.

To analyze the subsequent difference operators, the following asymptotic estimates are pertinent. Their proof will appear elsewhere.

THEOREM 1. *If n is a large parameter such that* $\arg n \to 0$ *as* $n \to \infty$, *the parameters a, λ, α_r and ρ_k are independent of n, and* $\beta = p + 1 - q \geqslant 3$, *then*

$$G_n(w) = \frac{\Gamma(n+1)}{\Gamma(n+\lambda)} G^{p+1,1}_{q+2,p+1}\left(w \,\middle|\, \begin{matrix} 1-n-\lambda, a+1-\rho_Q, n+1 \\ 0, a-\alpha_P \end{matrix}\right),$$
$$= \frac{1}{2\pi i}\int_{L_+} \frac{\Gamma(-s)\,\Gamma(-s+a-\alpha_P)(n+\lambda)_s\, w^s}{\Gamma(-s+a+1-\rho_Q)(n+1)_{-s}}\, ds, \tag{1.13}$$

$$\sim \sqrt{\frac{(2\pi)^{\beta-1}}{\beta}}\,[wN^2]^\tau \exp\left\{-\beta[wN^2]^{1/\beta} + \frac{w}{3}[wN^2]^{(3-\beta)/\beta} + O([w^5N^{10-4\beta}]^{1/\beta})\right\}$$
$$\times \{1 + O([1+|w|]^{2\Delta}[wN^2]^{-1/\beta})\},$$
$$wN^2 \to \infty, \qquad |\arg[wN^2]| < \pi[\beta+1],$$
$$w = o(N^{2\mu}), \qquad N^2 = n(n+\lambda),$$
$$\beta\tau = (a+\tfrac{1}{2})(\beta-1) + \sum_{j=1}^{q}\rho_j - \sum_{j=1}^{p}(1+\alpha_j),$$
$$\mu = \max\left(\frac{1}{3}, \frac{2}{5}\beta - 1\right), \qquad \Delta = \min\left(\frac{2}{3}, \frac{1}{2\beta-5}\right). \tag{1.14}$$

Moreover, if

$$\rho_k - \alpha_r \neq 0, -1, -2,\ldots; \quad k = 1,\ldots, q; \quad r = 0, 1,\ldots, p; \quad \alpha_0 = a,$$
$$\rho_k - \rho_j \neq \textit{an integer}, \qquad k \neq j, \tag{1.15}$$

then

$$L_{n,k}(w) = \frac{\Gamma(n+1)}{\Gamma(n+\lambda)}$$
$$\times\, G^{p+2,2\ 2}_{q+3,p+}\left(w \,\middle|\, \begin{matrix} 1-n-\lambda, a+1-\rho_k, a+1-\rho_Q, n+1 \\ 0, a-\alpha_P, a+1-\rho_k \end{matrix}\right), \tag{1.16}$$

$$= \frac{1}{2\pi i}\int_{L_+} \frac{\begin{pmatrix}\Gamma(-s)\,\Gamma(-s+a-\alpha_P)\ \Gamma(-s+a+1-\rho_k) \\ \times\,\Gamma(s-a+\rho_k)(n+\lambda)_s\, w^s\end{pmatrix}}{\Gamma(-s+a+1-\rho_Q)(n+1)_{-s}}\, ds,$$

$$\sim \frac{\Gamma(\rho_k-a)\,\Gamma(\rho_k-\alpha_P)}{\Gamma(\rho_k+1-\rho_Q)}[wN^2]^{a-\rho_k}\{1 + O([wN^2]^{-1})\}, \tag{1.17}$$
$$wN^2 \to \infty, \qquad |\arg[wN^2]| < \pi(\beta+2)/2.$$

That root of $[wN^2]^{1/\beta}$ *is chosen which has argument zero when* w, n *and* λ *are positive.*

For w independent of n and $|\arg w| < \pi\beta/2$, (1.14) is due to Wimp [6].

2. Main Results

THEOREM 2. *Let the parameters* a, λ, α_r *and* ρ_k *satisfy the conditions*

$$a = 0, 1,..., \text{ or } p; \qquad \lambda = 0, 1,..., \text{ or } p - a, \tag{2.1}$$

$$\rho_k - \alpha_r \neq 0, -1, -2,...; \quad k = 1,..., q; \quad r = 0, 1,..., p; \quad \alpha_0 = a, \tag{2.2}$$

$$\begin{aligned} \rho_k - \rho_j &\neq \text{an integer}, \quad k \neq j, \\ \alpha_r - \alpha_j &\neq \text{an integer}, \quad r \neq j, \end{aligned} \tag{2.3}$$

and set

$$H_n(v) = {}_{q+2}F_p\left(\begin{matrix}-n, n+\lambda, -a+\rho_Q \\ -a+1+\alpha_P\end{matrix}\,\middle|\, -v\right), \tag{2.4}$$

$$K_n(v) = \sum_{k=a}^{n} (-v)^k \sum_{j=0}^{n-k} \frac{(-n)_{k+j}\,(n+\lambda)_{k+j}\,(-a+\rho_Q)_{k+j}\,(\alpha_P)_j}{(-a+1+\alpha_P)_{k+j}\,(\rho_Q)_j\,(k+j)!\,j!}. \tag{2.5}$$

Then

$$\lim_{n\to\infty} \frac{K_n(v)}{H_n(v)} = \frac{\Gamma(\rho_Q)}{\Gamma(\alpha_P)}\, G^{1,p}_{p,q+1}\left(v^{-1}\,\middle|\, \begin{matrix} 1-\alpha_P \\ 0, 1-\rho_Q\end{matrix}\right),$$

$$v (\neq 0) \textit{ fixed}, \qquad |\arg v| < \pi, \qquad \beta = p + 1 - q > 2. \tag{2.6}$$

Moreover, the convergence is uniform on compact subsets of the open sector $|\arg v| < \pi$.

Proof. This will follow directly from Theorems 3 and 4 which analyze the asymptotic behaviour of $H_n(v)$ and $K_n(v)$, respectively. ∎

THEOREM 3. *Under the condition* (2.3), *there exist numbers* A_j, $j = 1,..., \beta$, *and* B_k, $k = 1,..., q$, *which are independent of* n *and* v *such that*

$$H_n(v) = {}_{q+2}F_p\left(\begin{matrix}-n, n+\lambda, -a+\rho_Q \\ -a+1+\alpha_P\end{matrix}\,\middle|\, ve^{i\pi}\right), \tag{2.7}$$

$$= \frac{\Gamma(n+1)\,\Gamma(-a+1+\alpha_P)}{\Gamma(n+\lambda)\,\Gamma(-a+\rho_Q)}$$

$$\times\ G^{1,q+1}_{q+2,p+1}\left(ve^{i\pi}\,\middle|\,\begin{matrix}1-n-\lambda,\ a+1-\rho_Q\,,\ n+1\\ 0,\ a-\alpha_P\end{matrix}\right), \tag{2.8}$$

$$= \frac{1}{2\pi i}\int_{L_+} \frac{\Gamma(-s)(-a+\rho_Q)_s\,(n+\lambda)_s\,(ve^{i\pi})^s}{(-a+1+\alpha_P)_s\,(n+1)_{-s}}\,ds,$$

$$= \sum_{j=1}^{\beta} A_j G_n(ve^{i\pi(\beta+2-2j)}) + \sum_{k=1}^{q} B_k L_{n,k}(ve^{i\pi(2\omega-\beta+1)}), \tag{2.9}$$

$$\sim A_1 G_n(ve^{i\pi\beta}),$$

$$\sim \frac{\Gamma(-a+1+\alpha_P)[vN^2]^\tau}{\Gamma(-a+\rho_Q)\,\sqrt{\beta}(2\pi)^{\beta-1}}\exp\{\beta[vN^2]^{1/\beta} + O([v^3N^{6-2\beta}]^{1/\beta})\}, \tag{2.10}$$

$$vN^2 \to \infty, \quad |\arg[vN^2]| < \pi, \quad \arg n \to 0, \quad v = o(N^{2\mu}),$$

where

$$\omega \text{ is the largest integer} \leqslant \frac{\beta}{2},$$

$$A_1 = (2\pi)^{1-\beta}\,e^{-i\pi\beta\tau}\,\frac{\Gamma(-a+1+\alpha_P)}{\Gamma(-a+\rho_Q)}, \tag{2.11}$$

and the other parameters are defined in (1.14).

Proof. From the partial fraction decomposition

$$\frac{\prod_{r=1}^{p}(y-a_r)}{\prod_{k=1}^{q}(y-b_k)} = \sum_{k=1}^{q}\frac{d_{k,m}\,y^m}{y-b_k} + \sum_{j=1}^{\beta} c_{j,m}\,y^{\beta-j},$$

$$c_{1,m} = 1, \qquad c_{\beta,m} = (-1)^{\beta-1}\frac{(a_P)_1}{(b_Q)_1}, \tag{2.12}$$

$$b_j \neq b_k\,, \qquad j \neq k, \qquad 0 < m < \beta = p+1-q,$$

with $y = e^{i\pi 2s}$, $a_r = e^{i2\pi(a-\alpha_r)}$, $b_k = e^{i2\pi(a-\rho_k)}$, it follows that

$$J_m(s) = \frac{\Gamma(s-a+\rho_Q)\,\Gamma(-s+a+1-\rho_Q)}{\Gamma(s-a+1+\alpha_P)\,\Gamma(-s+a-\alpha_P)},$$

$$= \sum_{k=1}^{q} d_{k,m}(2\pi)^{-\beta}\,e^{-i\pi(\frac{1}{2}+\beta\tau+a-\rho_k)}e^{i\pi s(2m-\beta)}\Gamma(s-a+\rho_k)\,\Gamma(-s+a+1-\rho_k)$$

$$+ \sum_{j=1}^{\beta} c_{j,m}(2\pi)^{1-\beta}\,e^{-i\pi\beta\tau}e^{i\pi s(\beta+1-2j)}. \tag{2.13}$$

The numbers A_j and B_k are then defined by

$$\frac{\Gamma(-a+1+\alpha_P)}{\Gamma(-a+\rho_Q)} J_\omega(s) = \sum_{k=1}^{\alpha} B_k e^{i\pi s(2\omega-\beta)} \Gamma(s-a+\rho_k)\,\Gamma(-s+a+1-\rho_k)$$
$$+ \sum_{j=1}^{\beta} A_j e^{i\pi s(\beta+1-2j)}. \tag{2.14}$$

In particular, A_1 is given by (2.11),

$$A_\beta = (2\pi)^{1-\beta}\, e^{i\pi\beta\tau}\, \frac{\Gamma(-a+1+\alpha_P)}{\Gamma(-a+\rho_Q)}, \tag{2.15}$$

and

$$B_k = e^{i\pi(a-\rho_k)(\beta-2\omega)}\, \frac{\Gamma(-a+1+\alpha_P)\,\Gamma(1+\rho_k-\rho_Q)\,\Gamma^*(-\rho_k+\rho_Q)}{\Gamma(-a+\rho_Q)\,\Gamma(1+\alpha_P-\rho_k)\,\Gamma(\rho_k-\alpha_P)}. \tag{2.16}$$

It follows from the integral definition of $H_n(v)$ as a G-function, with $L = L_+$, that

$$H_n(v) = \frac{\Gamma(-a+1+\alpha_P)}{\Gamma(-a+\rho_Q)}$$
$$\times \frac{1}{2\pi i}\int_{L_+} \frac{\Gamma(-s)\,\Gamma(-s+a-\alpha_P)(n+\lambda)_s\,(ve^{i\pi})^s}{\Gamma(-s+a+1-\rho_Q)(n+1)_{-s}}\, J_\omega(s)\, ds. \tag{2.17}$$

Equation (2.9) then follows when $J_\omega(s)$ is replaced by the expansion (2.14), and the resulting integrals are identified. For $|\arg v| < \pi$, the asymptotic expansion of all the resulting functions can be deduced from Theorem 1, and (2.10) follows directly. Note that for Theorems 1 and 3 to be valid, it is not necessary that n be a positive integer. ∎

THEOREM 4. *Under the conditions of Theorem* 2, *set*

$$F(v) = \sum_{r=1}^{p} v^{\alpha_r} F_r(v),$$
$$F_r(v) = \frac{(\alpha_P{}^*)_{-\alpha_r}}{(\rho_Q)_{-\alpha_r}}\, {}_{q+2}F_p\left(\begin{matrix}1,\ \alpha_r,\ \alpha_r+1-\rho_Q \\ \alpha_r+1-\alpha_P\end{matrix}\,\middle|\, (-1)^{\beta-1}\, v\right), \tag{2.18}$$

$$G_n(v) = \sum_{r=1}^{p} v^{a-\alpha_r} G_{n,r}(v) + \frac{\Gamma(a-\alpha_P)}{\Gamma(a+1-\rho_Q)}\, H_n((-1)^\beta\, v),$$

$$G_{n,r}(v) = \frac{\Gamma(-a+\alpha_r)\,\Gamma^*(\alpha_r-\alpha_P)(n+\lambda)_{a-\alpha_r}}{\Gamma(\alpha_r+1-\rho_Q)(n+1)_{-a+\alpha_r}} \tag{2.19}$$
$$\times F\left(\begin{matrix}1,\ -n+a-\alpha_r,\ n+\lambda+a-\alpha_r,\ \rho_Q-\alpha_r \\ a+1-\alpha_r,\ 1+\alpha_P-\alpha_r\end{matrix}\,\middle|\, (-1)^{\beta-1}\, v\right).$$

Then

$$K_n(v) = \sum_{r=1}^{p} \frac{\left(\begin{array}{l}\Gamma(-a+1+\alpha_P)\,\Gamma(1-\rho_Q+\alpha_r)\\ \qquad\qquad \times\, \Gamma(\rho_Q-\alpha_r)\, v^a G_{n,r}(ve^{i\pi\beta})\, F_r(v)\end{array}\right)}{\left(\begin{array}{l}\Gamma(-a+\rho_Q)\,\Gamma(1+\alpha_P-\alpha_r)\,\Gamma^*(\alpha_r-\alpha_P)\\ \qquad\qquad \times\, \Gamma(-a+\alpha_r)\,\Gamma(a+1-\alpha_r)\end{array}\right)},$$

$$\sim A_1 F(v)\, G_n(ve^{i\pi\beta}), \qquad n\to\infty, \qquad |\arg v| < \pi, \tag{2.20}$$

where A_1 is defined in (2.11).

Proof. Under the above parameter conditions, consider the integral

$$I_k = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{\Gamma(-s)(-a+\rho_Q)_{s+k}\,(\alpha_P)_s\,(n+\lambda)_{s+k}}{(-a+1+\alpha_P)_{s+k}\,(\rho_Q)_s\,(1)_{s+k}\,(n+1)_{-s-k}}\,ds,$$
$$k-a = 0, 1,\ldots, \tag{2.21}$$

where the contour from $-i\infty$ to $i\infty$ separates the poles of $\Gamma(-s)$ from those of $\Gamma(s+k-a+\rho_Q)\,\Gamma(s+\alpha_P)\,\Gamma(s+k+n+\lambda)$. The integral I_k is absolutely convergent for $\lambda-p-2+a-(k-a)(\beta-2) \leqslant \lambda-p-2+a<-1$. Evaluating I_k by the poles to the right and left of the contour, one obtains

$$\frac{(-1)^k\,(-n)_k\,(n+\lambda)_k\,(-a+\rho_Q)_k}{(-a+1+\alpha_P)_k\,k!}$$
$$\times\; {}_{p+q+2}F_{p+q+1}\left(\begin{matrix}-n+k,\, n+\lambda+k,\, -a+\rho_Q+k,\, \alpha_P\\ -a+1+\alpha_P+k,\, 1+k,\, \rho_Q\end{matrix}\,\middle|\, 1\right)$$
$$= \sum_{r=1}^{p} \frac{(-a+\rho_Q)_{k-\alpha_r}\,(\alpha_P{}^*)_{-\alpha_r}\,(n+\lambda)_{k-\alpha_r}}{(-a+1+\alpha_P)_{k-\alpha_r}\,(1)_{k-\alpha_r}\,(\rho_Q)_{-\alpha_r}\,(n+1)_{-k+\alpha_r}}$$
$$\times\, F\left(\begin{matrix}\alpha_r,\, a-k+\alpha_r-\alpha_P,\, -k+\alpha_r,\, 1+\alpha_r-\rho_Q,\, 1\\ 1-n-\lambda-k+\alpha_r,\, 1-\rho_Q+a-k+\alpha_r,\, 1+\alpha_r-\alpha_P,\, n+1-k+\alpha_r\end{matrix}\,\middle|\, 1\right). \tag{2.22}$$

Substituting this identity into the definition of $K_n(v)$, replacing k by $k+a$, and making use of the result

$$F_r(v)\, G_{n,r}(ve^{i\pi\beta})$$
$$= \frac{\left(\begin{array}{l}\Gamma(-a+\alpha_r)\,\Gamma(\alpha_r)\,\Gamma^*(\alpha_P-\alpha_r)\,\Gamma^*(\alpha_r-\alpha_P)\\ \qquad\qquad \times\, \Gamma(n+1)\,\Gamma(n+\lambda+a-\alpha_r)\,\Gamma(\rho_Q)\end{array}\right)}{\Gamma(\alpha_r+1-\rho_Q)\,\Gamma(\rho_Q-\alpha_r)\,\Gamma(n+1-a+\alpha_r)\,\Gamma(n+\lambda)\,\Gamma(\alpha_P)}$$
$$\times \sum_{k=0}^{\infty} \frac{(-n+a-\alpha_r)_k\,(n+\lambda+a-\alpha_r)_k\,(\rho_Q-\alpha_r)_k\,(-v)^k}{(a+1-\alpha_r)_k\,(1+\alpha_P-\alpha_r)_k}$$
$$\times\, F\left(\begin{matrix}-k-a+\alpha_r,\, -k+\alpha_r-\alpha_P,\, 1,\, \alpha_r,\, 1+\alpha_r-\rho_Q\\ -k+1-a+n+\alpha_r,\, -k+1-a-n-\lambda+\alpha_r,\, -k+1-\rho_Q+\alpha_r,\, 1+\alpha_r-\alpha_P\end{matrix}\,\middle|\, 1\right), \tag{2.23}$$

one obtains the first line of (2.20). A further computation using Theorem 2 with n replaced by $n - a + \alpha_r$ and v independent of n, shows that

$$G_{n,r}(ve^{i\pi\beta}) \sim \frac{\Gamma(-a+\alpha_r)\,\Gamma(1+a-\alpha_r)\,\Gamma^*(\alpha_r-\alpha_P)\,\Gamma(1+\alpha_P-\alpha_r)}{\Gamma(1-\rho_Q+\alpha_r)\,\Gamma(\rho_Q-\alpha_r)(2\pi)^{\beta-1}}$$
$$\times\, e^{-i\pi\beta\tau}v^{\alpha_r-a}G_n(ve^{i\pi\beta}), \qquad n\to\infty, \qquad |\arg v| < \pi, \tag{2.24}$$

and

$$K_n(v) \sim \frac{\Gamma(1-a+\alpha_P)\,e^{-i\pi\beta\tau}}{\Gamma(-a+\rho_Q)(2\pi)^{\beta-1}}\,G_n(ve^{i\pi\beta})\sum_{r=1}^{p} v^{\alpha_r}F_r(v), \tag{2.25}$$

which reduces to the last line of (2.20). Theorem 2 then follows directly. ∎

In the subsequent analysis, we will make use of the difference operators

$$\begin{aligned}\mathscr{M}_n(\gamma) = {}& \mathscr{U}_n(\lambda-p,0)\,\mathscr{U}_n(\lambda-p+1,-a+\alpha_1)\cdots\mathscr{U}_n(\lambda,-a+\alpha_p)\\ &- n(n+\lambda-p-1)\,\gamma\mathscr{E}^{-1}\mathscr{U}_n(\lambda-p+2,-a+\rho_1)\cdots\\ &\cdots\mathscr{U}_n(\lambda-p+q+1,-a+\rho_q)\,\mathscr{U}_n(\lambda-p+q+2)\cdots\mathscr{U}_n(\lambda),\end{aligned} \tag{2.26}$$

where

$$\begin{aligned}\mathscr{U}_n(\lambda,\mu) &= \frac{(n+\lambda-1)(n+\mu)}{2n+\lambda-1}\mathscr{E}^0 - \frac{n(n+\lambda-1-\mu)}{2n+\lambda-1}\mathscr{E}^{-1},\\ \mathscr{U}_n(\lambda) &= \lim_{\mu\to\infty}\frac{\mathscr{U}_n(\lambda,\mu)}{\mu} = \frac{(n+\lambda-1)}{2n+\lambda-1}\mathscr{E}^0 + \frac{n}{2n+\lambda-1}\mathscr{E}^{-1},\end{aligned} \tag{2.27}$$

and $\mathscr{E}^{-j}$ is the shift operator on n, i.e., $\mathscr{E}^{-j}y_n = y_{n-j}$. Clearly, $\mathscr{M}_n(\gamma)$ is of order $p+1$ and can be written in the form

$$\mathscr{M}_n(\gamma) = \sum_{j=0}^{p+1}[C_j(n,\lambda) + \gamma D_j(n,\lambda)]\,\mathscr{E}^{-j}. \tag{2.28}$$

Explicit expressions for the $C_j(n,\lambda)$ and $D_n(n,\lambda)$ can be deduced from [4, Chapter 12; 7]. A simple computation shows that

$$\mathscr{U}_n(\lambda,\mu)\left\{\frac{(n+\lambda)_s}{(n+1)_{-s}}\right\} = \frac{(n+\lambda-1)_s}{(n+1)_{-s}}(s+\mu), \tag{2.29}$$

and

$$\begin{aligned}\mathscr{M}_n(\gamma)\left\{\frac{(n+\lambda)_s}{(n+1)_{-s}}\right\} = {}& \frac{(n+\lambda-p-1)_s}{(n+1)_{-s}}\,s(s-a+\alpha_P)_1\\ &- \gamma\,\frac{(n+\lambda-p-1)_{s+1}}{(n+1)_{-s-1}}(s-a+\rho_Q)_1\,.\end{aligned} \tag{2.30}$$

Note that if s equal to an integer r, $(-1)^r(-n)_r(n+1)_{-r} = 1$.

THEOREM 5. *Under the conditions of Theorem 2, the functions $H_n(v)$, $K_n(v)$, $G_n(ve^{i\pi(\beta+2m)})$ and $L_{n,k}(ve^{i\pi(\beta-1+2m)})$, $k = 1,\ldots, q$, m an integer, all satisfy the linear difference equation of order $p + 1$,*

$$\mathscr{M}_n(v)\{y_n(v)\} = 0. \tag{2.31}$$

Moreover, if $|\arg v| < \pi$, the $p + 1$ functions $G_n(ve^{i\pi(\beta+2-2j)})$, $j = 1,\ldots,$ $\beta = p + 1 - q$, and $L_{n,k}(ve^{i\pi(2\omega-\beta+1)})$, $k = 1,\ldots, q$, form a basis $\mathscr{B}_{p+1}$ of (2.31). The parameter ω is defined in (2.11).

Proof. One can apply $\mathscr{M}_n(v)$ directly to the integral representations of $G_n(ve^{i\pi(\beta+2m)})$ and $L_{n,k}(ve^{i\pi(\beta-1+2m)})$ as given by (1.13) and (1.16). It then follows from (2.30) and the residue theorem that these functions satisfy (2.31) (see [8]). As $H_n(v)$ is a linear combination of these functions, it too satisfies (2.31). From the above and theorem 1, it is clear that $\mathscr{B}_{p+1}$ is a basis of (2.31). Finally, again making use of (2.30), we have

$$\mathscr{M}_n(\gamma)\left\{\sum_{k=a}^{n}\sum_{j=0}^{n-k}\frac{(-a+\rho_Q)_{k+j}\,(\alpha_P)_j\,(-n)_{k+j}\,(n+\lambda)_{k+j}}{(1-a+\alpha_P)_{k+j}\,(\rho_Q)_j\,(k+j)!\,j!}(-\gamma)^k\,(\gamma/v)^j\right\}$$

$$=\sum_{k=a}^{n}\sum_{j=0}^{n-k}\frac{(-n)_{k+j}\,(n+\lambda-p-1)_{k+j}\,(\alpha_P)_j\,(-a+\rho_Q)_{k+j}}{(\rho_Q)_j\,j!\,\Gamma(k+j)(1-a+\alpha_P)_{k+j-1}}(-\gamma)^k\,(\gamma/v)^j$$

$$-\sum_{k=a}^{n-1}\sum_{j=0}^{n-k-1}\frac{(-n)_{k+j+1}(n+\lambda-p-1)_{k+j+1}(\alpha_P)_j(-a+\rho_Q)_{k+j+1}}{(\rho_Q)_j\,j!\,(1-a+\alpha_P)_{k+j}\,\Gamma(k+j+1)}(-\gamma)^{k+1}\,(\gamma/v)^j,$$

$$=\sum_{j=0}^{n-a}\frac{(-n)_{j+a}\,(n+\lambda-p-1)_{j+a}\,\Gamma(1-a+\alpha_P)\,\Gamma(\rho_Q)}{\Gamma(a+j)\,j!\,\Gamma(-a+\rho_Q)\,\Gamma(\alpha_P)}(-\gamma)^a\,(\gamma/v)^j,$$

$$=\frac{\Gamma(1-a+\alpha_P)\,\Gamma(\rho_Q)(-n)_a\,(n+\lambda-p-1)_a}{\Gamma(-a+\rho_Q)\,\Gamma(\alpha_P)\,\Gamma(a)}(-\gamma)^a$$

$$\times\,{}_2F_1\left(\begin{matrix}-n+a,\,n+\lambda-p-1+a\\ a\end{matrix}\,\middle|\,\gamma/v\right),\qquad a\neq 0,$$

$$=(\alpha_P)_1\,(-n)(n+\lambda-p-1)(\gamma/v)\,{}_2F_1\left(\begin{matrix}-n+1,\,n+\lambda-p\\ 2\end{matrix}\,\middle|\,\gamma/v\right),$$

$$a = 0. \tag{2.32}$$

Thus

$$\mathscr{M}_n(v)\{K_n(v)\}$$

$$=(-1)^{n-a}\frac{n(n+\lambda-p-1)\,\Gamma(n+\lambda-p-1+a)\,\Gamma(1-a+\alpha_P)\,\Gamma(\rho_Q)}{\Gamma(n+1-a)\,\Gamma(\lambda+a-p)\,\Gamma(\alpha_p)\,\Gamma(-a+\rho_Q)}\gamma^a,$$

$$=0,\qquad \lambda+a-1-p \text{ a negative integer.}\quad\blacksquare \tag{2.33}$$

THEOREM 6. *Under the conditions of Theorem* 2,

$$K_n(v) = \sum_{j=1}^{\beta} C_j(v)\, G_n(ve^{i\pi(\beta+2-2j)}) + \sum_{k=1}^{q} D_k(v)\, L_{n,k}(ve^{i\pi(2\omega-\beta+1)}), \quad (2.34)$$

where

$$\begin{aligned} C_j(v) &= \sum_{k=1}^{j} (A_k - A_{k-1})\, F(ve^{-2\pi i(j-k)}), \qquad j = 1, 2,..., \beta - \omega, \\ &= \sum_{k=0}^{\beta-j} (A_{\beta-k} - A_{\beta+1-k})\, F(ve^{2\pi i(\beta+1-j-k)}), \quad j = \beta + 1 - \omega,..., \beta, \end{aligned} \quad (2.35)$$

$$D_k(v) = \frac{B_k E_k(v)}{\Gamma(\rho_k)\, \Gamma(1-\rho_k)}, \qquad E_k(v) = \frac{\Gamma(\rho_Q)}{\Gamma(\alpha_P)}\, G^{p+1,2}_{q+2,p+1}\left(ve^{i\pi} \,\middle|\, \begin{matrix} 1, \rho_k, \rho_Q \\ \rho_k, \alpha_P \end{matrix}\right), \quad (2.36)$$

and the A_k, B_k *are defined in* (2.14), $A_0 = A_{\beta+1} = 0$.

Proof. The existence of the expansion for $K_n(v)$ follows directly from Theorem 5, and only the identification of the coefficients remains.

We begin by noticing that the general relationship

$$\begin{aligned} &(2\pi i)\, G^{m,n}_{p,q+1}\left(w \,\middle|\, \begin{matrix} a_P \\ b_Q, \sigma \end{matrix}\right) \\ &= e^{i\pi\sigma} G^{m+1,n}_{p,q+1}\left(we^{-i\pi} \,\middle|\, \begin{matrix} a_P \\ \sigma, b_Q \end{matrix}\right) - e^{-i\pi\sigma} G^{m+1,n}_{p,q+1}\left(we^{i\pi} \,\middle|\, \begin{matrix} a_P \\ \sigma, b_Q \end{matrix}\right) \end{aligned} \quad (2.37)$$

implies that

$$L_{n,k}(we^{-2\pi i}) = e^{2\pi i\rho_k} L_{n,k}(w) + (-1)^{a+1}(2\pi i)\, e^{i\pi\rho_k} G_n(we^{-i\pi}), \quad (2.38)$$

and

$$E_k(we^{-2\pi i}) = e^{-2\pi i\rho_k} E_k(w) + (2\pi i)\, e^{-i\pi\rho_k} F(w). \quad (2.39)$$

Moreover, as $H_n(v)$ is a polynomial in v, $H_n(v) = H_n(ve^{-2\pi i})$. Combining this relationship with (2.9) and (2.38) one obtains the recursion relationship

$$\begin{aligned} 0 = &\sum_{j=0}^{\beta} (A_{j+1} - A_j)\, G_n(ve^{i\pi(\beta-2j)}) + (-1)^a (2\pi i)\, G_n(ve^{i\pi(2\omega-\beta)}) \sum_{k=1}^{q} e^{i\pi\rho_k} B_k \\ &+ \sum_{k=1}^{q} B_k(1 - e^{2\pi i\rho_k})\, L_{n,k}(ve^{i\pi(2\omega-\beta+1)}). \end{aligned} \quad (2.40)$$

This equation expresses $G_n(ve^{-i\pi\beta})$ in terms of the basis $\mathscr{B}_{p+1}$.

Furthermore, since

$$J_\omega(s) = (-1)^{a(p-q)+p} \frac{\Gamma(s + \rho_Q)\, \Gamma(-s + 1 - \rho_Q)}{\Gamma(s + \alpha_P)\, \Gamma(1 - s - \alpha_P)}, \tag{2.41}$$

it follows from (2.14) that

$$\frac{1}{2\pi i} \int_{L_-} \frac{\Gamma(-s)\, \Gamma(s + \alpha_P)}{\Gamma(s + \rho_Q)} J_\omega(s)\, v^{-s}\, ds$$

$$= \frac{(\rho_Q) \; -a}{(\alpha_P)\; 1 \; -a} \left\{ \sum_{j=1}^{\beta} A_j F(ve^{i\pi(2j-\beta-1)}) + \sum_{k=1}^{q} (-1)^a\, B_k E_k(ve^{-i\pi(2\omega-\beta+1)}) \right\}$$

$$= 0, \tag{2.42}$$

since the integrand of the integral has no poles interior to L_-.

As $G_n(ve^{i\pi\beta})$ is the dominant term of $K_n(v)$ as $n \to \infty$, Theorem 4 implies that $C_1(v) = A_1 F(v)$, or that (2.35) is valid for $j = 1$. The remaining coefficients are determined by the fact that $K_n(v)$ as a polynomial in v is invariant under the transformation v into $ve^{-2\pi i}$. When v is replaced by $ve^{-2\pi i}$ in (2.34), one obtains a linear expansion of $K_n(ve^{-2\pi i})$ in the $G_n(ve^{i\pi(\beta-2j)}), j = 1,..., \beta$ and $L_{n,k}(ve^{i\pi(2\omega-\beta-1)})$. If $G_n(ve^{-i\pi\beta})$ and $L_{n,k}(ve^{i\pi(2\omega-\beta-1)})$ are replaced using (2.40) and (2.38), respectively, one obtains a linear expansion of $K_n(ve^{-2\pi i})$ in terms of the basis $\mathscr{B}_{p+1}$ which must agree with the expansion of $K_n(v)$ in terms of the basis $\mathscr{B}_{p+1}$. In particular, if the coefficients of $G_n(ve^{i\pi\beta})$ are compared, one obtains $A_\beta C_1(v) = A_1 C_\beta(ve^{-2\pi i})$ or $C_\beta(v) = A_\beta F(ve^{2\pi i})$. More generally,

$$C_j(v) = C_{j-1}(ve^{-2\pi i}) + (A_j - A_{j-1})\, F(v), \qquad j = 1, 2,..., \beta, \quad j \neq 1 + \beta - \omega, \tag{2.43}$$

and

$$C_{1+\beta-\omega}(v) = C_{\beta-\omega}(ve^{-2\pi i}) + (-1)^{a+1} (2\pi i) \sum_{k=1}^{q} e^{i\pi\rho_k} D_k(ve^{-2\pi i})$$

$$+ (A_{1+\beta-\omega} - A_{\beta-\omega})\, F(v) + (-1)^a (2\pi i) F(v) \sum_{k=1}^{q} e^{i\pi\rho_k} B_k . \tag{2.44}$$

Equating the coefficients of $L_{n,k}(ve^{i\pi(2\omega-\beta+1)})$, we obtain

$$D_k(v) = e^{2\pi i\rho_k} D_k(ve^{-2\pi i}) + \frac{(-2\pi i)\, e^{i\pi\rho_k} B_k}{\Gamma(\rho_k)\, \Gamma(1 - \rho_k)} F(v). \tag{2.45}$$

Letting $j = 2, 3,..., \beta - \omega$ in (2.43), one obtains by induction the first line of (2.35). Letting $j = \beta, \beta - 1,..., 2 + \beta - \omega$ in (2.43), one obtains the second line of (2.35). This evaluates all the $C_j(v)$. To evaluate the $D_k(v)$, set

$$D_k(v) = \frac{B_k E_k(v)}{\Gamma(\rho_k)\, \Gamma(1 - \rho_k)} + v^{\rho_k} D_k^{\#}(v). \tag{2.46}$$

We wish to show that $D_k^{\#}(v) \equiv 0$, $k = 1,..., q$. Then (2.39) and (2.45) imply that

$$D_k^{\#}(v) = D_k^{\#}(ve^{-2\pi i}). \tag{2.47}$$

Making use of (2.42) with v replaced by $ve^{i\pi(2\omega-\beta+1)}$ and $ve^{i\pi(2\omega-\beta-1)}$, (2.44) can then be written in the form

$$\sum_{k=1}^{q} e^{-i\pi\rho_k} v^{\rho_k} D_k^{\#}(v) = 0. \tag{2.48}$$

Replacing v by $ve^{2\pi i r}$, r an integer, and using (2.47) this last equation can be written in the form

$$\sum_{k=1}^{q} e^{i\pi\rho_k(2r-1)} v^{\rho_k} D_k^{\#}(v) = 0. \tag{2.49}$$

Letting $r = 1, 2,..., q$, one obtains a homogeneous system of q equations in the q unknowns $v^{\rho_k} D_k^{\#}(v)$. As

$$\begin{vmatrix} x_1 & ,..., x_q \\ x_1^3 & ,..., x_q^3 \\ x_1^{2q-1} & ,..., x_q^{2q-1} \end{vmatrix} = \prod_{k=1}^{q} x_j \cdot \prod_{1 \leqslant j < k \leqslant q} (x_k^2 - x_j^2), \tag{2.50}$$

it is clear that with $x_k = e^{i\pi\rho_k}$, the determinent of the above system is not zero under the condition (2.3). Hence $D_k^{\#}(v) = 0$, $k = 1,..., q$, establishing (2.36). ▌

COROLLARY. *Under the conditions of Theorem* 2,

$$\begin{aligned} S_n(v) &= F(v)\, H_n(v) - K_n(v), \\ &= \sum_{j=2}^{\beta} [A_j F(v) - C_j(v)]\, G_n(ve^{i\pi(\beta+2-2j)}) \\ &\quad + \sum_{k=1}^{q} \frac{B_k e^{i\pi\rho_k}}{2\pi i} [E_k(ve^{-2\pi i}) - E_k(v)]\, L_{n,k}(ve^{i\pi(2\omega-\beta+1)}), \end{aligned} \tag{2.51}$$

$$A_jF(v) - C_j(v) = \sum_{k=1}^{j-1} A_k[F(ve^{-i2\pi(j-1-k)}) - F(ve^{-i2\pi(j-k)})],\ j = 2,\dots,\beta - \omega,$$

$$= \sum_{k=j}^{\beta} A_k[F(ve^{i2\pi(k-j)}) - F(ve^{i2\pi(k+1-j)})],\ j = \beta + 1 - \omega,\dots,\beta, \tag{2.52}$$

$$S_n(v)/H_n(v) = O(\exp\{-\sigma(v) \mid vN^2 \mid^{1/\beta}\},$$

$$\sigma(v) = 2\beta \cos\left[\frac{\pi(\beta-2)}{2\beta}\right] \cos\left[\frac{2\mid \arg v\mid + \pi(\beta-2)}{2\beta}\right], \tag{2.53}$$

$$> 0, \qquad \mid \arg v \mid < \pi.$$

3. Remarks

Remark 1. It appears that Theorem 2 is valid when condition (2.3) is weakened. For when this condition is violated, the exponential terms, $G_n(ve^{i\pi(\beta+2-2j)})$, and their coefficients in the expansions (2.9) and (2.34) of $H_n(v)$ and $K_n(v)$ remain well defined. It appears that the limiting form of the sum of the $L_{n,k}(ve^{i\pi(2\omega-\beta+1)})$ terms is $O([N^2v]^\sigma [\log (vN^2)]^m)$, $n \to \infty$, where σ is a constant and m is a positive integer. This would be subdominant to the exponential terms G_n. Condition (2.2) is suspect also.

Remark 2. If λ does not satisfy condition (2.1), or γ is not equal to v (say $\gamma = rv$, $0 < r < 1$), then $K_n^a(v, \gamma)$ would be a solution of the nonhomogeneous difference Eq. (2.32). It appears that the more involved analysis of this nonhomogeneous equation would still lead to the results of Theorem 2.

Remark 3. Preliminary computations indicate that for $\beta = 2$, the asymptotic representation for $L_{n,k}(w)$, (1.17), remains valid, and

$$G_n(w) = \frac{\Gamma(n+1)}{\Gamma(n+\lambda)} G_{q+2,q+2}^{q+2,1}\left(w \,\middle|\, \begin{matrix} 1-n-\lambda, a+1-\rho_Q, n+1 \\ 0, a-\alpha_{Q+1} \end{matrix}\right), \tag{3.1}$$

$$\sim \sqrt{\pi}\,[wN^2]^\tau\,[1+w]^{-\tau-\lambda/2}\,e^{-N\xi}\{1+O(N^{-1})\},$$

$$wN^2 \to \infty, \qquad (1+w)\,N^2 \to \infty, \qquad N^2 = n(n+\lambda), \tag{3.2}$$

$$\mid \arg[wN^2]\mid < 3\pi, \qquad \mid \arg[(1+w)\,N^2]\mid < \pi, \qquad \cosh \xi = 1 + 2w,$$

and τ is defined in (1.14). For $q = 0$, these results have been established by

Watson [9]. A rigorous derivation of the general case would imply that the results of section 2 are also valid for $\beta = 2$.

Remark 4. Clearly the range of validity, $|\arg v| < \pi$, in Theorem 2 is optimal as $F(v)$ is multiple-valued and takes on different values at $ve^{i\pi}$ and $ve^{-i\pi}$—a behaviour that rational approximations cannot be expected to mimic. This optimality is also reflected in the fact that asymptotically the zeros of $H_n(v)$ lie in a sector $|\arg(ve^{-i\pi})| < \epsilon$, $\epsilon > 0$.

References

1. H. S. WALL, "Analytic Theory of Continued Fractions," Van Nostrand, Princeton, New Jersey, 1948.
2. A. ERDÉLYI, W. MAGNUS, F. OBERHETTINGER AND F. G. TRICOMI, "Higher Transcendental Functions," Vol. 1, McGraw–Hill, New York, 1953.
3. Y. L. LUKE, "The Special Functions and their Approximations," Vol. 1. Academic Press, New York, 1969.
4. Y. L. LUKE, "The Special Functions and their Approximations," Vol. 2. Academic Press, New York, 1969.
5. J. L. FIELDS, "Rational approximations to generalized hypergeometric functions" *Math. Comp.* **19** (1965), 606–624.
6. J. WIMP, "The asymptotic representation of a class of *G*-functions for large parameter," *Math. Comp.* **21** (1967), 639–646.
7. J. L. FIELDS, Y. L. LUKE AND J. WIMP, Recursion formulae for generalized hypergeometric functions, *J. Approximation Theory* **1** (1968), 137–166.
8. J. WIMP AND J. L. FIELDS, On the factorization of a class of difference operators. *Bull. Amer. Math. Soc.* **74** (1968), 1068–1071.
9. G. N. WATSON, Asymptotic expansions of hypergeometric functions, *Trans. Cambridge Phil. Soc.* **22** (1918), 277–308.